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Small divisors of Bernoulli sums

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ABSTRACT

Let $\varepsilon = \{\varepsilon_i, i \geqslant 1\}$ be a sequence of independent Bernoulli random variables ($\mathbf{P}\{\varepsilon_i = 0\} = \mathbf{P}\{\varepsilon_i = 1\} = 1/2$) with basic probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Consider the sequence of partial sums $B_n = \varepsilon_1 + \cdots + \varepsilon_n$, $n = 1, 2, \ldots$. We obtain an asymptotic estimate for the probability $\mathbf{P}\{P^-(B_n) > \zeta\}$ for $\zeta \leqslant n^{c/\log\log n}$, c a positive constant.

1. INTRODUCTION AND RESULTS

Let $\beta = \{\beta_i, i \geqslant 1\}$ be a Bernoulli sequence, and let $B_n = \beta_1 + \cdots + \beta_n$, $n = 1, 2, \ldots,$ be the sequence of associated partial sums. Let $(\Omega, \mathcal{A}, \mathbf{P})$ denote the underlying basic probability space. We study the probability $\mathbf{P}\{P^-(B_n) > y\}$ that the smallest prime divisor of B_n exceeds y, where $y \leqslant y(n)$ and $y(n)$ tends to infinity with n. We used the notation $P^-(m)$ to denote the smallest divisor of m.

The work is thereby continuing the study of the distribution value of divisors undertaken in previous papers [5,6,8] which we briefly recall. In [8] (see Theorems 1.1 and 2.1), precise estimates of the probability $\mathbf{P}\{d|S_n\}$ are established next applied to prove the existence of another form of the functional equation of the Zeta-Riemann function. Here S_n, $n = 1, 2 \ldots$, denotes the sequence of partial sums of independent spin random variables ($\mathbf{P}\{\varepsilon_i = \pm 1\} = 1/2$). In [5] (see Lemma 2.1 and estimates (2.5)), partial estimates of $\mathbf{P}\{d|B_n\}$ are established

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and next combined with a probabilistic trick to study the convergence of the series of divisors functions $\sum_{n=1}^{\infty} a_n d(n,\mathcal{D})$, $\sum_{n=1}^{\infty} a_n d_2(n,\mathcal{D})$, where $d(n,\mathcal{D}) = \sum_{d\in\mathcal{D}, d\leqslant\sqrt{n}} \chi(d|n)$, $d_2(n,\mathcal{D}) = \sum_{d,\delta\in\mathcal{D},[d,\delta]\leqslant\sqrt{n}} \chi([d,\delta]|n)$, $\mathcal{D}$ being a prescribed set of integers. We also refer to [7] and [9] for applications to the study of Riemann sums, and [1] to LIL and SLLN results related to additive arithmetic functions. We refer to [6] for the investigation of an extremal divisor case ($n = m^2$, $d = m$, $m \to \infty$). Here to the contrary, we are concerned with the case $d \to \infty$ with n but $d \ll n$. The main result of the paper can be stated as follows

Theorem I. *There exists a positive real $c > 0$ and constants C_0, ζ_0 such that for n large enough and $\zeta_0 \leqslant \zeta \leqslant n^{c/\log\log n}$, we have the following estimate*

$$\left|\mathbf{P}\{P^-(B_n) > \zeta\} - \frac{e^{-\gamma}}{\log\zeta}\right| \leqslant \frac{C_0}{\log^2\zeta},$$

where γ is Euler's constant.

The proof is based on a uniform estimate for the value distribution of Bernoulli sums. Consider the elliptic Theta function

$$\Theta(d,m) = \sum_{\ell\in\mathbf{Z}} e^{im\pi\frac{\ell}{d} - \frac{m\pi^2\ell^2}{2d^2}}.$$

Theorem II. *We have the following uniform estimate*:

$$\sup_{2\leqslant d\leqslant n}\left|\mathbf{P}\{d|B_n\} - \frac{\Theta(d,n)}{d}\right| = \mathcal{O}\big((\log n)^{5/2} n^{-3/2}\big). \tag{1.1}$$

Also

$$\left|\mathbf{P}\{d|B_n\} - \frac{1}{d}\right| \leqslant \begin{cases} C\Big((\log n)^{5/2} n^{-3/2} + \frac{1}{d} e^{-\frac{n\pi^2}{2d^2}}\Big) & \text{if } d \leqslant \sqrt{n}, \\ \frac{C}{\sqrt{n}} & \text{if } \sqrt{n} \leqslant d \leqslant n. \end{cases} \tag{1.2}$$

Further, for any $\alpha > 0$

$$\sup_{d<\pi\sqrt{\frac{n}{2\alpha\log n}}}\left|\mathbf{P}\{d|B_n\} - \frac{1}{d}\right| = \mathcal{O}_\varepsilon\big(n^{-\alpha+\varepsilon}\big) \quad (\forall\varepsilon > 0) \tag{1.3}$$

and for any $0 < \rho < 1$,

$$\sup_{d<(\pi/\sqrt{2})n^{(1-\rho)/2}}\left|\mathbf{P}\{d|B_n\} - \frac{1}{d}\right| = \mathcal{O}_\varepsilon\big(e^{-(1-\varepsilon)n^\rho}\big) \quad (\forall 0 < \varepsilon < 1). \tag{1.4}$$

It follows from (1.2) that $\lim_{n\to\infty} \mathbf{P}\{d|B_n\} = 1/d$, and

$$(1.5) \qquad \left|\mathbf{P}\{d|B_n\} - \frac{1}{d}\right| \leqslant C\frac{d}{n} \quad \text{if } 2 \leqslant d \leqslant \sqrt{n}.$$

In particular

$$(1.6) \qquad \sup_{2\leqslant d\leqslant\sqrt{n}} d\mathbf{P}\{d|B_n\} \leqslant C.$$

Estimate (1.4) yields a strong variation of the speed of convergence (as $n \to \infty$) of $\mathbf{P}\{d|B_n\}$ to its limit $1/d$, when switching from the case $d \leqslant \sqrt{n}$ to the case $d \leqslant n^\theta$, $\theta < 1/2$.

2. PROOF OF THEOREM II

By writing that

$$(2.1) \qquad d\delta_{d|B_n} = \sum_{j=0}^{d-1} e^{2i\pi\frac{j}{d}B_n},$$

we obtain after integration

$$(2.2) \qquad \mathbf{P}\{d|B_n\} = \frac{1}{d}\sum_{j=0}^{d-1} e^{i\pi n\frac{j}{d}}\left(\cos\frac{\pi j}{d}\right)^n.$$

The summands are of the form $e^{inx}\cos^n x$. As

$$e^{in(\pi-x)}\cos^n(\pi - x) = (-1)^n e^{-inx}(-1)^n\cos^n x = e^{-inx}\cos^n x,$$

we have in fact, by distinguishing the case d even from the case d odd

$$(2.3) \qquad \mathbf{P}\{d|B_n\} = \frac{1}{d} + \frac{2}{d}\sum_{1\leqslant j<d/2}\left(\cos\pi n\frac{j}{d}\right)\left(\cos\frac{\pi j}{d}\right)^n.$$

Now, the principle of the proof will consist in comparing the sum $\sum_{1\leqslant j<d/2}(\cos\pi n\frac{j}{d})(\cos\frac{\pi j}{d})^n$ with the following one

$$\sum_{1\leqslant j<d/2}\left(\cos\pi n\frac{j}{d}\right)e^{-n\frac{\pi^2 j^2}{2d^2}}.$$

According to the reduction operated in (2.3), we only have to work in the first quadrant. Let $\alpha > \alpha' > 0$. Let

$$(2.4) \qquad \varphi_n = \left(\frac{2\alpha\log n}{n}\right)^{1/2}, \qquad \tau_n = \frac{\sin\varphi_n/2}{\varphi_n/2}.$$

We assume n sufficiently large, say $n \geqslant n_0$, for τ_n to be greater than $(\alpha'/\alpha)^{1/2}$. Consider two sectors

$$A_n =]0, \varphi_n[, \qquad A'_n = \left[\varphi_n, \frac{\pi}{2}\right[.$$

If $\frac{\pi j}{d} \in A'_n$, then $|\cos \frac{\pi j}{d}| \leqslant \cos\varphi_n$. And $|\cos \frac{\pi j}{d}|^n \leqslant (\cos\varphi_n)^n \leqslant e^{-2n\sin^2(\varphi_n/2)}$. As $2n\sin^2(\varphi_n/2) = 2n(\varphi_n/2)^2\tau_n^2 \geqslant \alpha' \log n$, we deduce

$$(2.5) \qquad \sum_{1\leqslant j<d/2:\ \frac{\pi j}{d}\in A'_n} \left|\cos\frac{\pi j}{d}\right|^n \leqslant dn^{-\alpha'}.$$

Now, if $d < \pi\sqrt{\frac{n}{2\alpha \log n}}$, we deduce that

$$\frac{\pi j}{d} > \frac{\pi j}{\pi\sqrt{\frac{n}{2\alpha\log n}}} = \sqrt{\frac{2\alpha\log n}{n}} = \varphi_n,$$

and so $\{1 \leqslant j < d/2\colon \frac{\pi j}{d} \in A_n\} = \emptyset$.

Therefore, in view of (2.3), (2.5), for each $\alpha > 0$

$$(2.6) \qquad \sup_{d<\pi\sqrt{\frac{n}{2\alpha\log n}}} \left|\mathbf{P}\{d|B_n\} - \frac{1}{d}\right| = \mathcal{O}_\varepsilon\left(n^{-\alpha+\varepsilon}\right) \quad (\forall\varepsilon > 0).$$

Now, let $0 < \rho < 1$. In a similar fashion

$$(2.7) \qquad \sup_{d<(\pi/\sqrt{2})n^{(1-\rho)/2}} \left|\mathbf{P}\{d|B_n\} - \frac{1}{d}\right| = \mathcal{O}_\varepsilon\left(e^{-(1-\varepsilon)n^\rho}\right) \quad (\forall 0 < \varepsilon < 1),$$

which are respectively estimates (1.3) and (1.4) of Theorem I. Indeed, consider the modified sectors

$$\widetilde{A}_n =]0, \psi_n[, \qquad \widetilde{A}'_n = \left[\psi_n, \frac{\pi}{2}\right[,$$

where

$$(2.8) \qquad \psi_n = \left(\frac{2n^\rho}{n}\right)^{1/2}, \qquad \widetilde{\tau}_n = \frac{\sin\psi_n/2}{\psi_n/2}.$$

Let also $0 < \varepsilon < 1$, and suppose n sufficiently large for $\widetilde{\tau}_n$ to be greater than $\sqrt{1-\varepsilon}$. Exactly as before, if $\frac{\pi j}{d} \in \widetilde{A}'_n$, then $|\cos\frac{\pi j}{d}| \leqslant \cos\psi_n$, so that $|\cos\frac{\pi j}{d}|^n \leqslant (\cos\psi_n)^n \leqslant e^{-2n\sin^2(\psi_n/2)}$. And $2n\sin^2(\psi_n/2) = 2n(\psi_n/2)^2\tau_n^2 = n^\rho\tau_n^2 \geqslant (1-\varepsilon)n^\rho$. We deduce

$$(2.9) \qquad \sum_{1\leqslant j<d/2:\ \frac{\pi j}{d}\in\widetilde{A}'_n} \left|\cos\frac{\pi j}{d}\right|^n \leqslant de^{-(1-\varepsilon)n^\rho}.$$

For the same reasons as before, this sum is the only term appearing in (2.3), when $d < \pi\sqrt{\frac{n}{2n^\rho}}$, since and so $\{1 \leqslant j < d/2 \colon \frac{\pi j}{d} \in \widetilde{A}_n\} = \emptyset$.

Now assume $\alpha > \alpha' > 3/2$. Apart from (2.5), the inequality $\varphi_n = (\frac{2\alpha \log n}{n})^{1/2} \leqslant \frac{\pi j}{d} < \frac{\pi}{2}$ implies that

$$\sum_{1 \leqslant j < d/2 \colon \frac{\pi j}{d} \in A_n'} e^{-n\frac{\pi^2 j^2}{2d^2}} \leqslant dn^{-\alpha}. \tag{2.10}$$

Now consider the contribution of the terms for which $\frac{\pi j}{d} \in A_n$. We proceed as follows: if

$$D := \sum_{1 \leqslant j < d/2 \colon \frac{\pi j}{d} \in A_n} \left(\cos \pi n \frac{j}{d}\right)\left(\cos^n \frac{\pi j}{d} - e^{-n\frac{\pi^2 j^2}{2d^2}}\right),$$

then by using the elementary inequality: $|e^u - e^v| \leqslant |u - v|$ for $u, v \leqslant 0$

$$|D| \leqslant n \sum_{1 \leqslant j < d/2 \colon \frac{\pi j}{d} \in A_n} \left|\log \cos \frac{\pi j}{d} + \frac{\pi^2 j^2}{2d^2}\right|.$$

Since $\log(1 - 2\sin^2(x/2)) = -x^2/2 + \mathcal{O}(x^4)$ near 0, we deduce

$$\begin{aligned} |D| &\leqslant Cn \sum_{1 \leqslant j < d/2 \colon \frac{\pi j}{d} \in A_n} \left(\frac{j}{d}\right)^4 \leqslant \frac{Cn}{d^4} \sum_{j le \frac{d}{\pi}(\frac{2\alpha \log n}{n})^{1/2}} j^4 \\ &\leqslant C_\alpha d (\log n)^{5/2} n^{-3/2}. \end{aligned} \tag{2.11}$$

Combining (2.5), (2.10) and (2.11) shows that

$$\begin{aligned} &\left|\sum_{1 \leqslant j < d/2} \left(\cos \pi n \frac{j}{d}\right)\left(\cos^n \frac{\pi j}{d} - e^{-n\frac{\pi^2 j^2}{2d^2}}\right)\right| \\ &\leqslant |D| + \sum_{1 \leqslant j < d/2 \colon \frac{\pi j}{d} \in A_n'} \left(\left|\cos \frac{\pi j}{d}\right|^n + e^{-n\frac{\pi^2 j^2}{2d^2}}\right) \\ &\leqslant C_\alpha d (\log n)^{5/2} n^{-3/2} + dn^{-\alpha'} + dn^{-\alpha} \\ &\leqslant C_\alpha d (\log n)^{5/2} n^{-3/2}. \end{aligned} \tag{2.12}$$

Dividing both sides by d, and reporting next the obtained estimate into (2.3) gives

$$\left|\mathbf{P}\{d | B_n\} - \frac{1}{d} - \frac{2}{d} \sum_{1 \leqslant j < d/2} \left(\cos \pi n \frac{j}{d}\right) e^{-n\frac{\pi^2 j^2}{2d^2}}\right| \leqslant C_\alpha (\log n)^{5/2} n^{-3/2}. \tag{2.13}$$

As

$$\frac{1}{d}+\frac{2}{d}\sum_{1\leqslant j<d/2}\left(\cos\pi n\frac{j}{d}\right)e^{-n\frac{\pi^2 j^2}{2d^2}}=\frac{1}{d}\sum_{|j|<d/2}e^{\pi n\frac{j}{d}}e^{-n\frac{\pi^2 j^2}{2d^2}},$$

we obtained

$$\text{(2.14)}\qquad \sup_{2\leqslant d\leqslant n}\left|\mathbf{P}\{d|B_n\}-\frac{1}{d}\sum_{|j|<d/2}e^{i\pi n\frac{j}{d}-n\frac{\pi^2 j^2}{2d^2}}\right|=\mathcal{O}\big((\log n)^{5/2}n^{-3/2}\big).$$

Now consider the remainder $r:=\sum_{j\geqslant d/2}e^{-n\frac{\pi^2 j^2}{2d^2}}$. By using the triangle inequality

$$\text{(2.15)}\qquad \left|\mathbf{P}\{d|B_n\}-\frac{\Theta(d,n)}{d}\right|\leqslant C(\log n)^{5/2}n^{-3/2}+\frac{2r}{d}.$$

– If $d=2$,

$$\text{(2.16a)}\ r=\sum_{j=1}^{\infty}e^{-n\frac{\pi^2 j^2}{8}}\leqslant e^{-\frac{\pi^2 n}{8}}+\sum_{j=2}^{\infty}\int_{\frac{\pi(j-1)}{2\sqrt{2}}}^{\frac{\pi j}{2\sqrt{2}}}e^{-nx^2}dx=e^{-\frac{\pi^2 n}{8}}+\int_{\frac{\pi}{2\sqrt{2}}}^{\infty}e^{-nx^2}dx$$

$$\stackrel{(x=\frac{y}{\sqrt{2n}})}{=}e^{-\frac{\pi^2 n}{8}}+\int_{\frac{\pi\sqrt{n}}{2}}^{\infty}e^{-y^2/2}\frac{dy}{\sqrt{2n}}\leqslant Ce^{-\frac{\pi^2 n}{8}}.$$

It is also worth noticing (referee's observation), since for n large enough the ratio of two consecutive terms of the sequence $e^{-n\frac{\pi^2 j^2}{8}}$ is $\leqslant 1/2$, that the sum $\sum_{j=1}^{\infty}e^{-n\frac{\pi^2 j^2}{8}}$ is at most twice of its first term.

– Now if $d\geqslant 3$, then

$$\frac{\frac{d}{2}-1}{d}\geqslant\frac{\frac{d}{2}-\frac{d}{3}}{d}=\frac{1}{6}.$$

Therefore

$$\text{(2.16)}\qquad r\leqslant\sum_{j\geqslant d/2}^{\infty}\int_{\frac{\pi(j-1)}{\sqrt{2}d}}^{\frac{\pi j}{\sqrt{2}d}}e^{-nx^2}dx\leqslant\int_{\frac{\pi(\frac{d}{2}-1)}{\sqrt{2}d}}^{\infty}e^{-nx^2}dx$$

$$\leqslant\int_{\frac{\pi}{6\sqrt{2}}}^{\infty}e^{-nx^2}dx\stackrel{(x=\frac{y}{\sqrt{2n}})}{=}\int_{\frac{\pi\sqrt{n}}{6}}^{\infty}e^{-y^2/2}\frac{dy}{\sqrt{2n}}$$

$$\leqslant Ce^{-\frac{\pi^2 n}{72}},$$

a bound which is in turn valid for all integers $d \geqslant 2$ and $n \geqslant 1$. To get (1.1), it suffices to incorporate these estimates into (2.14).

We now shall need the useful estimate below. Let a be any positive real. Then,

$$\sum_{H=1}^{\infty} e^{-aH^2} \leqslant \begin{cases} 3e^{-a} & \text{if } a \geqslant 1, \\ 3/\sqrt{a} & \text{if } a \leqslant 1. \end{cases} \tag{2.17}$$

Indeed

$$\begin{aligned} \sum_{H=1}^{k} e^{-aH^2} &\leqslant e^{-a} + \sum_{H=2}^{k} e^{-aH^2} \leqslant e^{-a} + \int_1^{\infty} e^{-au^2} du \\ &\overset{u=v/\sqrt{2a}}{=} e^{-a} + \frac{1}{\sqrt{2a}} \int_{\sqrt{2a}}^{\infty} e^{-v^2/2} dv \\ &\leqslant e^{-a}\left(1 + \sqrt{\frac{\pi}{2a}}\right), \end{aligned}$$

where we used the elementary bound $\int_x^{\infty} e^{-t^2/2} dt \leqslant \sqrt{\frac{\pi}{2}} e^{-x^2/2}$ valid for all $x \geqslant 0$. Thus $\sum_{H=1}^{\infty} e^{-aH^2} \leqslant e^{-a}(1 + \sqrt{\pi/2}) \leqslant 3e^{-a}$ if $a \geqslant 1$, and $\sum_{H=1}^{\infty} e^{-aH^2} \leqslant (1/\sqrt{a})(1 + \sqrt{\frac{\pi}{2}}) \leqslant 3/\sqrt{a}$, if $a \leqslant 1$. Hence (2.17).

Therefore

$$\left|\frac{\Theta(d,n)}{d} - \frac{1}{d}\right| \leqslant \begin{cases} \dfrac{C}{d} e^{-\frac{n\pi^2}{2d^2}} & \text{if } d \leqslant \sqrt{n}, \\ \dfrac{C}{\sqrt{n}} & \text{if } \sqrt{n} \leqslant d \leqslant n. \end{cases}$$

And so

$$\left|\mathbf{P}\{d | B_n\} - \frac{1}{d}\right| \leqslant \begin{cases} C\left((\log n)^{5/2} n^{-3/2} + \dfrac{1}{d} e^{-\frac{n\pi^2}{2d^2}}\right) & \text{if } d \leqslant \sqrt{n}, \\ \dfrac{C}{\sqrt{n}} & \text{if } \sqrt{n} \leqslant d \leqslant n, \end{cases} \tag{2.18}$$

which is (1.2).

The proof is now complete. □

Remark. The following estimate is easily deduced from the last part of the above proof.

$$\left|\mathbf{P}\{d | B_n\} - \frac{1}{d}\right| \leqslant C_\alpha \frac{\log^{5/2} n}{n^{3/2}} + \frac{2}{d} \sum_{1 \leqslant j \leqslant \frac{d}{\pi}\sqrt{\frac{2\alpha \log n}{n}}} e^{-n\frac{\pi^2 j^2}{2d^2}}. \tag{2.19}$$

Indeed,

$$
\begin{aligned}
\left|\mathbf{P}\{d|B_n\}-\frac{1}{d}\right| &\leqslant \frac{2}{d}\Bigg|\sum_{\substack{1\leqslant j<d/2\\ \frac{\pi j}{d}\in A_n'}}\left(\cos\pi n\frac{j}{d}\right)\left(\cos\frac{\pi j}{d}\right)^n\Bigg|\\
&\quad+\frac{2}{d}\Bigg|\sum_{\substack{1\leqslant j<d/2\\ \frac{\pi j}{d}\in A_n}}\left(\cos\pi n\frac{j}{d}\right)\left(\cos\frac{\pi j}{d}\right)^n\Bigg|\\
&\leqslant C_\varepsilon n^{-\alpha+\varepsilon}+\frac{2}{d}\Bigg|\sum_{\substack{1\leqslant j<d/2\\ \frac{\pi j}{d}\in A_n}}\left(\cos\pi n\frac{j}{d}\right)\left(\cos\frac{\pi j}{d}\right)^n\Bigg|\\
&\leqslant C_\varepsilon n^{-\alpha+\varepsilon}+\frac{2}{d}|D|+\frac{2}{d}\Bigg|\sum_{\substack{1\leqslant j<d/2\\ \frac{\pi j}{d}\in A_n}}\left(\cos\pi n\frac{j}{d}\right)e^{-n\frac{\pi^2 j^2}{2d^2}}\Bigg|\\
&\leqslant C_\varepsilon n^{-\alpha+\varepsilon}+C_\alpha\frac{\log^{5/2}n}{n^{3/2}}+\frac{2}{d}\sum_{1\leqslant j\leqslant\frac{d}{\pi}\sqrt{\frac{2\alpha\log n}{n}}}e^{-n\frac{\pi^2 j^2}{2d^2}}\\
&\leqslant C_\alpha\frac{\log^{5/2}n}{n^{3/2}}+\frac{2}{d}\sum_{1\leqslant j\leqslant\frac{d}{\pi}\sqrt{\frac{2\alpha\log n}{n}}}e^{-n\frac{\pi^2 j^2}{2d^2}}.
\end{aligned}
$$

3. PROOF OF THEOREM I

Fix some $0<\theta<1/2$ and some integer $D\geqslant 2$ such that $2\log 2D-(2+\frac{1}{D})\geqslant 1$. Let $3\leqslant Y\leqslant n$. Choose

$$(3.1)\qquad K=\left\lfloor\frac{2D}{\theta}\log\log Y+1\right\rfloor.$$

It will be later (see before (3.16)) necessary to have $K\geqslant e^7$, which is fulfilled for say, $Y\geqslant Y_0$, Y_0 depending on θ and D only; so that we in turn consider $Y_0\leqslant Y\leqslant n$. Set also

$$(3.2)\qquad m=\lfloor D\log\log Y\rfloor.$$

Then

$$\frac{2m}{K}\leqslant\frac{2D\log\log Y}{\frac{2D}{\theta}\log\log Y}=\theta.$$

And so

$$(3.3)\qquad Y^{\frac{2m}{K}}\leqslant n^\theta.$$

Let

$$(3.4)\qquad p_1<p_2<\cdots<p_H,$$

the sequence of primes less or equal to $Y^{1/K}$. By using Poincaré inequality (see for instance [3, p. 66])

$$\text{(3.5)} \qquad \mathbf{P}\{\exists 1 \leqslant i \leqslant H:\ p_i | B_n\} \leqslant \sum_{k=1}^{2m-1} (-1)^{k-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant H} \mathbf{P}\{p_{i_1} \cdots p_{i_k} | B_n\}.$$

To estimate the probability $\mathbf{P}\{p_{i_1} \dots p_{i_k} | B_n\}$ we use Theorem II. Let $0 < \varepsilon < 1$ be fixed. In view of (1.4), and since (3.3) implies $p_{i_1} \dots p_{i_k} < (\pi/\sqrt{2})n^\theta$ for $1 \leqslant i_1 < \cdots < i_k \leqslant H$, $k \leqslant 2m - 1$; there exists some positive real ϑ depending on θ only, such that for n large enough,

$$\text{(3.6)} \qquad \left| \mathbf{P}\{p_{i_1} \dots p_{i_k} | B_n\} - \frac{1}{p_{i_1} \dots p_{i_k}} \right| \leqslant e^{-n^\vartheta}.$$

From (3.5) and (3.6) follows

$$\text{(3.7)} \qquad \mathbf{P}\{\exists 1 \leqslant i \leqslant H:\ p_i | B_n\} \leqslant \sum_{k=1}^{2m-1} (-1)^{k-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant H} \frac{1}{p_{i_1} \dots p_{i_k}} + T,$$

where

$$\text{(3.8)} \qquad |T| \leqslant e^{-n^\vartheta} \sum_{k=1}^{2m-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant H} 1.$$

Letting $\xi_1, \dots, \xi_H$ be a sequence of independent random variables taking only the values 0 or 1 and such that $\mathbf{P}\{\xi_i = 1\} = \frac{1}{p_i}$, we see from Poincaré equality (see again [3, p. 66]) that

$$\text{(3.9)} \qquad \begin{aligned} \mathbf{P}\{\exists i:\ 1 \leqslant i \leqslant H,\ \xi_i = 1\} &= \sum_{k=1}^{H} (-1)^{k-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant H} \frac{1}{p_{i_1} \dots p_{i_k}} \\ &= 1 - \prod_{1 \leqslant i \leqslant H} \left(1 - \frac{1}{p_i}\right). \end{aligned}$$

And so,

$$\text{(3.10)} \qquad \sum_{k=1}^{2m-1} (-1)^{k-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant H} \frac{1}{p_{i_1} \dots p_{i_k}} = 1 - \prod_{1 \leqslant i \leqslant H} \left(1 - \frac{1}{p_i}\right) + S,$$

with

$$\text{(3.11)} \qquad |S| \leqslant \sum_{k \geqslant 2m} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant H} \frac{1}{p_{i_1} \dots p_{i_k}}.$$

Thereby,

$$(3.12)\qquad \mathbf{P}\{P^-(B_n)\leqslant Y^{1/K}\}\leqslant 1-\prod_{1\leqslant i\leqslant H}\left(1-\frac{1}{p_i}\right)+S+T.$$

As for any positive reals $b_1,\dots,b_H$

$$(3.13)\qquad \left(\sum_{i=1}^{H}b_i\right)^k=\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant H}b_{i_1}\dots b_{i_k}\geqslant k!\sum_{1\leqslant i_1<\cdots<i_k\leqslant H}b_{i_1}\dots b_{i_k}$$

we have

$$(3.14)\qquad \sum_{1\leqslant i_1<\cdots<i_k\leqslant H}\frac{1}{p_{i_1}\dots p_{i_k}}\leqslant\frac{1}{k!}\left(\sum_{i\leqslant H}\frac{1}{p_i}\right)^k.$$

And since (for any natural r, any complex number y) we have the elementary bound $|e^y-(1+\frac{y}{1!}+\cdots+\frac{y^r}{1!})|\leqslant\frac{|y|^{r+1}}{(r+1)!}e^{|y|}$, we deduce

$$(3.15)\qquad |S|\leqslant\sum_{k\geqslant 2m}\frac{1}{k!}\left(\sum_{i\leqslant H}\frac{1}{p_i}\right)^k\leqslant\frac{\left(\sum_{i\leqslant H}\frac{1}{p_i}\right)^{2m}}{(2m)!}e^{\sum_{i\leqslant H}\frac{1}{p_i}}.$$

Recall now the useful estimate $\sum_{p\leqslant x}\frac{1}{p}\leqslant\log\log x+6$ valid for any $x\geqslant 3$ (see [4, Theorem 9, p. 17]). Thus

$$|S|\leqslant\frac{(\log\log Y^{1/K}+6)^{2m}}{(2m)!}e^{(\log\log Y^{1/K}+6)}.$$

But $K\geqslant e^7$ implies

$$\begin{aligned}\log\log Y^{1/K}+6&\leqslant\log\frac{\log Y}{e^7}+6=\log\log Y-1\\&=\frac{D\log\log Y-D}{D}\leqslant\frac{D\log\log Y-1}{D}\leqslant\frac{m}{D},\end{aligned}$$

so that

$$(3.16)\qquad |S|\leqslant\left(\frac{m}{D}\right)^{2m}\frac{1}{(2m)!}e^{(\log\log Y^{1/K}+6)}.$$

Using then Cesàro–Buchner estimate ([2, inequality (6), p. 183]), we deduce from (3.16) and definition of m, that

$$\begin{aligned}(3.17)\qquad |S|&\leqslant\left(\frac{m}{D}\right)^{2m}\frac{1}{(2m)^{2m}}\frac{1}{2\sqrt{\pi m}}e^{2m+\log\log Y^{1/K}+6}\\&\leqslant\left(\frac{1}{2D}\right)^{2m}\frac{1}{2\sqrt{\pi m}}e^{2m+\log\log Y+6}\end{aligned}$$

$$\leqslant \left(\frac{1}{2D}\right)^{2m} \frac{1}{2\sqrt{\pi m}} e^{m(2+\frac{1}{D})+7} = \frac{e^7}{2\sqrt{\pi m}} e^{-m[2\log 2D-(2+\frac{1}{D})]}$$

$$\leqslant Ce^{-m} \leqslant Ce^{-D\log\log Y} = C\left(\frac{1}{\log Y}\right)^D.$$

According to Mertens formula (see [4, Theorem 11, p. 18] for instance), for any $x \geqslant 2$,

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1+\mathcal{O}\left(\frac{1}{\log x}\right)\right). \tag{3.18}$$

Thus for some universal constant C,

$$\prod_{1\leqslant i\leqslant H}\left(1-\frac{1}{p_i}\right) = \prod_{p\leqslant Y^{1/K}}\left(1-\frac{1}{p}\right) \geqslant \frac{Ke^{-\gamma}}{\log Y}\left(1-\frac{CK}{\log Y}\right). \tag{3.19}$$

We now estimate T. This term has a small order. Using (3.13) we get

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant H} 1 \leqslant \frac{H^k}{k!},$$

and so

$$\begin{aligned} |T| &\leqslant e^{-n^\vartheta}\sum_{k=1}^{2m-1}\frac{H^k}{k!} \leqslant e^{-n^\vartheta}H^{2m} \leqslant e^{-n^\vartheta}Y^{2m/K} \\ &\leqslant e^{-n^\vartheta}n^\theta \leqslant e^{-n^\vartheta/2}, \end{aligned} \tag{3.20}$$

for n large enough. By reporting now this estimate as well as preceding estimates (3.17) and (3.19) into (3.12), we arrive to

$$\mathbf{P}\{P^-(B_n) \leqslant Y^{1/K}\} \leqslant 1 - \frac{Ke^{-\gamma}}{\log Y} + C\left(\frac{K}{\log Y}\right)^2 + \left(\frac{1}{\log Y}\right)^D + e^{-n^\vartheta/2}. \tag{3.21}$$

Finally, using the definition of K and since $D \geqslant 2$, for n any large enough

$$\mathbf{P}\{P^-(B_n) \leqslant Y^{1/K}\} \leqslant 1 - \frac{2De^{-\gamma}\log\log Y}{\theta\log Y} + C_0\left(\frac{2De^{-\gamma}\log\log Y}{\theta\log Y}\right)^2,$$

where C_0 is an absolute constant (one can take $C_0 = 2C$ where C is the constant in (3.19)). Or else,

$$\mathbf{P}\{P^-(B_n) > e^{\frac{\theta\log Y}{2D\log\log Y}}\} \geqslant \frac{2De^{-\gamma}\log\log Y}{\theta\log Y} - C_0\left(\frac{2De^{-\gamma}\log\log Y}{\theta\log Y}\right)^2. \tag{3.22}$$

Put

$$\zeta = e^{\frac{\theta\log Y}{2D\log\log Y}}.$$

Then $\zeta \leqslant e^{\frac{\theta \log n}{2D \log\log n}} := n^{c/\log\log n}$ and (3.21) means

$$(3.23)\qquad \mathbf{P}\{P^-(B_n) > \zeta\} \geqslant \frac{e^{-\gamma}}{\log \zeta} - \frac{C_0}{\log^2 \zeta}.$$

But we have also prepared the arguments to prove the upper bound part. By using again Poincaré inequalities, next (3.6), (3.14), (3.18) we get for any integer n large enough

$$(3.24)\qquad \begin{aligned} \mathbf{P}\{\exists p \leqslant Y^{1/K}\colon\ p|B_n\} &\geqslant \sum_{k=1}^{2m} (-1)^{k-1} \sum_{1\leqslant i_1<\cdots<i_k\leqslant H} \mathbf{P}\{p_{i_1}\cdots p_{i_k}|B_n\} \\ &\geqslant \sum_{k=1}^{2m} (-1)^{k-1} \sum_{1\leqslant i_1<\cdots<i_k\leqslant H} \frac{1}{p_{i_1}\cdots p_{i_k}} - T' \\ &\geqslant 1 - \prod_{1\leqslant i\leqslant H}\left(1-\frac{1}{p_i}\right) - S' - T' \\ &\geqslant 1 - \frac{Ke^{-\gamma}}{\log Y}\left(1+\frac{CK}{\log Y}\right) - S' - T', \end{aligned}$$

where

$$(3.25)\qquad \begin{aligned} |S'| &\leqslant \sum_{k\geqslant 2m+1} \sum_{1\leqslant i_1<\cdots<i_k\leqslant H} \frac{1}{p_{i_1}\cdots p_{i_k}}, \\ |T'| &\leqslant e^{-n^\vartheta} \sum_{k=1}^{2m} \sum_{1\leqslant i_1<\cdots<i_k\leqslant H} 1. \end{aligned}$$

From (3.18) we get $|S'| \leqslant (\frac{1}{\log Y})^D$, whereas in a same manner as to get (3.21) we have $|T'| \leqslant e^{-n^\vartheta/2}$. Reporting this into (3.25) also provides the following estimate

$$(3.26)\qquad \mathbf{P}\{\exists p \leqslant Y^{1/K}\colon\ p|B_n\} \geqslant 1 - \frac{Ke^{-\gamma}}{\log Y}\left(1+\frac{CK}{\log Y}\right) - \left(\frac{K}{\log Y}\right)^D - e^{-n^\vartheta/2}.$$

In view of the definition of K, for n any large enough

$$(3.27)\quad \mathbf{P}\{P^-(B_n) \leqslant e^{\frac{\theta \log Y}{2D \log\log Y}}\} \geqslant 1 - \frac{2De^{-\gamma}\log\log Y}{\theta \log Y} - C_0\left(\frac{2De^{-\gamma}\log\log Y}{\theta \log Y}\right)^2.$$

Thereby

$$\mathbf{P}\{P^-(B_n) > e^{\frac{\theta \log Y}{2D \log\log Y}}\} \leqslant \frac{2De^{-\gamma}\log\log Y}{\theta \log Y} + C_0\left(\frac{2De^{-\gamma}\log\log Y}{\theta \log Y}\right)^2$$

or

$$(3.28)\qquad \mathbf{P}\{P^-(B_n) > \zeta\} \leqslant \frac{e^{-\gamma}}{\log \zeta} + \frac{C_0}{\log^2 \zeta}.$$

The proof is achieved by combining (3.23) with (3.27). $\square$

Final note. In the course of the probabilistic proof of Theorem I we gave, we did not use estimate (1.4), and we believe that Theorem I should be improved by using either a suitable modification of the main arguments or elaborated sieve methods.

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